Classification of Eight-vertex Solutions of The Colored Yang-Baxter Equation

Shi-kun WANG

Abstract: In this paper all eight-vertex type solutions of the colored Yang-Baxter equation dependent on spectral as well as color parameter are given. It is proved that they are composed of three groups of basic solutions, three groups of their degenerate forms and two groups of trivial solutions up to five solution transformations. Moreover, all non-trivial solutions can be classified into two types called Baxter type and Free-Fermion type.

§0. **Introduction**

The Yang-Baxter or triangle equation which first appeared in Refs. 1-3 plays a prominent role in many branches of physics, for instance, in factorized S-matrices [4], exactly solvable models of statistical physics [5], complete integrable quantum and classical systems [6], quantum groups [7], conformal field theory and link invariants [8-11], to name just a few. In view of the importance of the Yang-Baxter equation, much attention has been directed to the search for solutions of the equation.

Colored Yang-Baxter equation dependent on spectral as well as color parameters is a generalization of usual Yang-Baxter equation. It has also attracted a lot of research interest (see Refs. 12-16) to find exact solutions for this type of Yang-Baxter equation. This is because the colored Yang-Baxter equation concerns free Fermion model in magnetic field, multi-variable invariants of links and representations of quantum algebras and so on (see Refs. 17-21).

The eight-vertex type solution of the colored Yang-Baxter equation has been investigated previously in Refs. 12, 17 and 20. In Ref. 17 Fan and Wu first provided a single relation between the eight vertex weights in the general eight-vertex model by the pfaffian or dimer method, the so-called free-fermion condition. Based on this work, V.V Bazhanov and Yu. G. Stroganov obtained a eight-vertex solution for the colored Yang-Baxter equation in Ref. 12, devoted to the eight-vertex free fermion model on a plane lattice. In Ref. 20 J. Murakami gave another eight-vertex solution in discussing multi-variable invariants of links. These are only two eight-vertex solutions for the colored Yang-Baxter equation we have known up to now.

The main theme of this paper is to give and classify all eight-vertex solutions of the colored Yang-Baxter equation. The way to find the solution is from a computer algebra method given by Wu in Ref. 22. Moreover, a theorem in the Ref. 22 can prove that all solutions can be obtained by Wu's method. The paper is organized as follows. In section one we will review the colored Yang-Baxter equation which in fact is a matrix equation. If the equation is expressed in components form it is composed of 28 polynomial equations in the eight-vertex case. In this section we will first introduce the symmetries or solution transformation for this system of equations and some definitions of Hamiltonian coefficient, initial value condition, unitary condition and non-trivial gauge solution of the colored Yang-Baxter equation. Using the symmetries we can simply the system of equations as 12 polynomial equations. In section two we will apply the computer algebra method to the 12 polynomial equations to get the algebraic curves and differential equations satisfied by the eight-vertex type solution of the colored Yang-Baxter equation. In this section we will also give two relations satisfied by Hamiltonian coefficient which will play an important role in classification of eight-vertex solutions of the colored Yang-Baxter equation. Based on the second section, in the third section we will construct all non-trivial gauge eight-vertex type solutions of the colored Yang-Baxter equation and classify them into two types called Baxter and Free-Fermion type. The fourth section is devoted to general solutions and the relation between Hamiltonian coefficients and spin-chain Hamiltonian. In the third and fourth sections we will also show the two solutions appeared in Refs. 12 and 20 are special cases of the general solutions obtained in this paper.

In this paper symbolic computation will be applied to accomplish some tedious computations and the results obtained by computer calculations will be denoted by the symbol * .

§1. **Colored Yang-Baxter equation, its symmetry and initial condition**

By colored Yang-Baxter equation we mean the following matrix equation

$$\begin{gathered}\check{R}_{12}(u,\xi,\eta)\check{R}_{23}(u+v,\xi,\lambda)\check{R}_{12}(v,\eta,\lambda)=\check{R}_{23}(v,\eta,\lambda)\check{R}_{12}(u+v,\xi,\lambda)\check{R}_{23}(u,\xi,\eta),\\ \check{R}_{12}(u,\xi,\eta)=\check{R}(u,\xi,\eta)\otimes E,\qquad \check{R}_{23}(u,\xi,\eta)=E\otimes\check{R}(u,\xi,\eta),\end{gathered}\tag{1.1}$$

where $\check{R}(u,\xi,\eta)$ is a matrix function of N^2-dimension of u,ξ and η, E is the unit matrix of order N and $\otimes$ means tensor product of two matrices. u,v are called spectral parameters and ξ, η colored parameters. If the matrix is independent of colored parameters, then

the colored Yang-Baxter equation (1.1) will become as the usual Yang-Baxter equation. If it is independent of spectral parameter then (1.1) will be reduced to the pure colored Yang-Baxter equation.

$$\check{R}_{12}(\xi,\eta)\check{R}_{23}(\xi,\lambda)\check{R}_{12}(\eta,\lambda) = \check{R}_{23}(\eta,\lambda)\check{R}_{12}(\xi,\lambda)\check{R}_{23}(\xi,\eta). \tag{1.2}$$

For the colored Yang-Baxter equation (1.1), the main interest in the paper is to discuss the solutions with the following form

$$\check{R}(u,\xi,\eta) = \begin{pmatrix} R_{11}^{11}(u,\xi,\eta) & 0 & 0 & R_{22}^{11}(u,\xi,\eta) \\ 0 & R_{12}^{12}(u,\xi,\eta) & R_{21}^{12}(u,\xi,\eta) & 0 \\ 0 & R_{12}^{21}(u,\xi,\eta) & R_{21}^{21}(u,\xi,\eta) & 0 \\ R_{11}^{22}(u,\xi,\eta) & 0 & 0 & R_{22}^{22}(u,\xi,\eta) \end{pmatrix}. \tag{1.3}$$

The eight weight functions in (1.3) are denoted by

$$a_1(u,\xi,\eta) = R_{11}^{11}(u,\xi,\eta), \quad a_5(u,\xi,\eta) = R_{21}^{12}(u,\xi,\eta),$$

$$a_2(u,\xi,\eta) = R_{12}^{12}(u,\xi,\eta), \quad a_6(u,\xi,\eta) = R_{12}^{21}(u,\xi,\eta),$$

$$a_3(u,\xi,\eta) = R_{21}^{21}(u,\xi,\eta), \quad a_7(u,\xi,\eta) = R_{22}^{11}(u,\xi,\eta),$$

$$a_4(u,\xi,\eta) = R_{22}^{22}(u,\xi,\eta), \quad a_8(u,\xi,\eta) = R_{11}^{22}(u,\xi,\eta).$$

We call the solution (1.3) eight-vertex type solution if it satisfies in addition $a_i(u,\xi,\eta) \not\equiv 0$ $(i = 1, 2, \cdots, 8)$. Further, we only consider the solutions which are meromorphic functions of u and ξ, η. For notational simplicity, throughout this paper let

$$u_i = a_i(u,\xi,\eta), \quad v_i = a_i(v,\eta,\lambda), \quad w_i = a_i(u+v,\xi,\lambda), \qquad i = 1, 2, \cdots, 8.$$

For eight-vertex type solutions, the matrix equation (1.1) is equivalent to the following 28 equations:

$$\left.\begin{aligned} u_7w_3v_8 - u_8w_2v_7 &= 0, \\ u_7w_8v_3 - u_8w_7v_2 &= 0, \\ u_2w_3v_2 - u_3w_2v_3 &= 0, \\ u_2w_8v_7 - u_3w_7v_8 &= 0, \end{aligned}\right\} \tag{1.4a}$$

$$\left.\begin{aligned}
u_1w_5v_2+u_7w_8v_6-v_2w_1u_5-v_5w_2u_3&=0,\\
u_1w_1v_7+u_7w_3v_4-v_7w_5u_5-v_1w_7u_3&=0,\\
u_2w_6v_1+u_5w_7v_8-v_6w_1u_2-v_3w_2u_6&=0,\\
u_1w_2v_1+u_7w_4v_8-v_2w_1u_2-v_5w_2u_6&=0,\\
u_1w_7v_5+u_7w_6v_3-v_7w_5u_2-v_1w_7u_6&=0,\\
u_1w_7v_2+u_7w_6v_6-v_1w_1u_7-v_7w_2u_4&=0,
\end{aligned}\right\}\tag{1.4b}$$

$$\left.\begin{aligned}
u_4w_6v_2+u_7w_8v_5-v_2w_4u_6-v_6w_2u_3&=0,\\
u_4w_4v_7+u_7w_3v_1-v_7w_6u_6-v_4w_7u_3&=0,\\
u_2w_5v_4+u_6w_7v_8-v_5w_4u_2-v_3w_2u_5&=0,\\
u_4w_2v_4+u_7w_1v_8-v_2w_4u_2-v_6w_2u_5&=0,\\
u_4w_7v_6+u_7w_5v_3-v_7w_6u_2-v_4w_7u_5&=0,\\
u_4w_7v_2+u_7w_5v_5-v_4w_4u_7-v_7w_2u_1&=0,
\end{aligned}\right\}\tag{1.4c}$$

$$\left.\begin{aligned}
u_1w_5v_3+u_8w_7v_6-v_3w_1u_5-v_5w_3u_2&=0,\\
u_1w_1v_8+u_8w_2v_4-v_8w_5u_5-v_1w_8u_2&=0,\\
u_3w_6v_1+u_5w_8v_7-v_6w_1u_3-v_2w_3u_6&=0,\\
u_1w_3v_1+u_8w_4v_7-v_3w_1u_3-v_5w_3u_6&=0,\\
u_1w_8v_5+u_8w_6v_2-v_8w_5u_3-v_1w_8u_6&=0,\\
u_1w_8v_3+u_8w_6v_6-v_1w_1u_8-v_8w_3u_4&=0,
\end{aligned}\right\}\tag{1.4d}$$

$$\left.\begin{aligned}
&u_4w_6v_3+u_8w_7v_5-v_3w_4u_6-v_6w_3u_2=0,\\
&u_4w_4v_8+u_8w_2v_1-v_8w_6u_6-v_4w_8u_2=0,\\
&u_3w_5v_4+u_6w_8v_7-v_5w_4u_3-v_2w_3u_5=0,\\
&u_4w_3v_4+u_8w_1v_7-v_3w_4u_3-v_6w_3u_5=0,\\
&u_4w_8v_6+u_8w_5v_2-v_8w_6u_3-v_4w_8u_5=0,\\
&u_4w_8v_3+u_8w_5v_5-v_4w_4u_8-v_8w_3u_1=0.
\end{aligned}\right\} \tag{1.4e}$$

Assume $\check{R}(u,\xi,\eta)$ is a solution of (1.1). Having carefully studied the system of equations (1.4), we find there are five symmetries for eight-vertex type solutions of the colored Yang-Baxter equation (1.1).

(A) Symmetry of interchanging indices.

The system of equations (1.4) is invariant if we interchange the two sub-indices 2 and 3 as well as the two sub-indices 7 and 8 or the sub-indices 1 and 4 as well as the two sub-indices 5 and 6.

(B) The scaling symmetry.

Multiplication of the solution $\check{R}(u,\xi,\eta)$ by an arbitrary function $g(u,\xi,\eta)$ is still a solution of the colored Yang-Baxter equation (1.1).

(C) Symmetry of weight functions

If the weight functions $a_2(u,\xi,\eta), a_3(u,\xi,\eta), a_7(u,\xi,\eta)$ and $a_8(u,\xi,\eta)$ are replaced by the new weight functions

$$\begin{aligned}
&\bar{a}_2(u,\xi,\eta)=\frac{N(\xi)}{N(\eta)}a_2(u,\xi,\eta), \quad \bar{a}_3(u,\xi,\eta)=\frac{N(\eta)}{N(\xi)}a_3(u,\xi,\eta),\\
&\bar{a}_7(u,\xi,\eta)=s\,N(\eta)N(\xi)a_7(u,\xi,\eta), \quad \bar{a}_8(u,\xi,\eta)=\frac{1}{s\,N(\eta)N(\xi)}a_8(u,\xi,\eta)
\end{aligned}$$

respectively or $a_5(u,\xi,\eta)$ and $a_6(u,\xi,\eta)$ by $-a_5(u,\xi,\eta)$ and $-a_6(u,\xi,\eta)$, where $N(\xi)$ is an arbitrary function of colored parameter and s is a complex constant, the new matrix $\check{R}(u,\xi,\eta)$ is still a solution of (1.1).

(D) Symmetry of spectral parameter.

If we take a new spectral parameter $\bar{u}=\mu u$ where μ is a complex constant independent of spectral and colored parameters , the new matrix $\check{R}(u',\xi,\eta)$ is still a solution of (1.1).

(E) Symmetry of color parameters. If we take new colored parameters $\zeta=f(\xi)$, $\theta=f(\eta)$, where $f(\xi)$ is an arbitrary function , then the new matrix $\check{R}(u,\zeta,\theta)$ is also a solution of (1.1).

The five symmetries (A)–(E) are called solution transformations A–E of eight-vertex type solutions of the colored Yang-Baxter equation (1.1) respectively.

Dividing both sides of the third of (1.4a) by $a_2(u,\xi,\eta)a_2(u+v,\xi,\lambda)a_2(v,\eta,\lambda)$, we get

$$f(u+v,\xi,\lambda)=f(u,\xi,\eta)f(v,\eta,\lambda), \tag{1.5}$$

where $f(u,\xi,\eta)=a_3(u,\xi,\eta)/a_2(u,\xi,\eta)$. Putting $u=v=\eta=0$ in (1.5) we have

$$f(0,\xi,\lambda)=f(0,\xi,0)f(0,0,\lambda).$$

Substituting this formula into the one obtained by taking $u=v=\xi=0$ in (1.5) we get

$$f(0,0,\lambda)=f(0,0,\eta)f(0,\eta,\lambda)=f(0,0,\eta)f(0,\eta,0)f(0,0,\lambda).$$

This means

$$f(0,\eta,0)f(0,0,\eta)=1.$$

Otherwise, it is easy to show that $f(u,\xi,\eta)=0$, i. e. $a_3(u,\xi,\eta)=0$. Therefore

$$f(0,\xi,\eta)=\frac{M(\xi)}{M(\eta)}, \tag{1.6}$$

where $M(\xi)=f(0,\xi,0)$. On the other hand, if we differentiate both sides of (1.5) with respect to the spectral variable v and then set $v=0$, $\lambda=\eta$, then

$$f'(u,\xi,\eta)=f'(0,\eta,\eta)f(u,\xi,\eta) \tag{1.7a}$$

holds, where the dot means derivative with respect to u and the simple formula

$$\frac{dG(u+v)}{dv}|_{v=0}=\frac{dG(u)}{du},$$

for any function $G(u)$, is used. Similarly, one also has

$$f'(v,\xi,\lambda)=f'(0,\xi,\xi)f(v,\xi,\lambda) \tag{1.7b}$$

if we differentiate both sides of (1.5) with respect to u and then set $u=0$ and $\eta=\xi$. The two formulas above imply $f'(0,\xi,\xi)$ is a constant independent of colored parameter ξ. Hence

$$f(u,\xi,\eta)=\frac{M(\xi)}{M(\eta)}\exp(\nu u), \tag{1.8}$$

where ν is a complex constant.

From the group of equations (1.4a) we have

$$f(u,\xi,\eta)h(v,\eta,\lambda) = h(u+v,\xi,\lambda), \tag{1.9a}$$

$$f(u,\xi,\eta)f(v,\eta,\lambda) = f(u+v,\xi,\lambda), \tag{1.9b}$$

$$h(u,\xi,\eta) = h(u+v,\xi,\lambda)f(v,\eta,\lambda), \tag{1.9c}$$

$$h(u,\xi,\eta) = f(u+v,\xi,\lambda)h(v,\eta,\lambda), \tag{1.9d}$$

where $f(u,\xi,\eta) = a_3(u,\xi,\eta)/a_2(u,\xi,\eta)$, $h(u,\xi,\eta) = a_8(u,\xi,\eta)/a_7(u,\xi,\eta)$. If we let $v = 0$ and $\lambda = 0$ in (1.9d) then

$$h(u,\xi,\eta) = f(u,\xi,0)h(0,\eta,0).$$

Substituting this into (1.9c) and using (1.9b), one obtains

$$f(u,\xi,0)h(0,\eta,0) = f(u,\xi,0)f(v,0,0)h(0,\lambda,0)f(v,\eta,\lambda)$$

or

$$h(0,\eta,0) = f(v,0,0)h(0,\lambda,0)f(v,\eta,\lambda).$$

This formula implies $\nu = 0$ in (1.8) and then one can obtain

$$f(u,\xi,\eta) = \frac{M(\xi)}{M(\eta)}, \quad h(u,\xi,\eta) = l\,M(\xi)M(\eta)$$

where l is a complex constant independent of spectral and colored parameters.

So, up to the solution transformation **B** and **C** one can assume

$$a_3(u,\xi,\eta) = a_2(u,\xi,\eta) = 1, \qquad a_8(u,\xi,\eta) = a_7(u,\xi,\eta)$$

without losing generality. In the case (1.4a), (1.4b), (1.4c) and (1.4d) can be simplified to the following 12 equations,

$$
\begin{aligned}
&a_5(v,\eta,\lambda)+a_5(u,\xi,\eta)a_1(u+v,\xi,\lambda)\\
&-a_1(u,\xi,\eta)a_5(u+v,\xi,\lambda)-a_7(u,\xi,\eta)a_7(u+v,\xi,\lambda)a_6(v,\eta,\lambda)=0,\\
&a_7(u+v,\xi,\lambda)a_1(v,\eta,\lambda)+a_5(u,\xi,\eta)a_5(u+v,\xi,\lambda)a_7(v,\eta,\lambda)\\
&-a_1(u,\xi,\eta)a_1(u+v,\xi,\lambda)a_7(v,\eta,\lambda)-a_7(u,\xi,\eta)a_4(v,\eta,\lambda)=0,\\
&a_6(u,\xi,\eta)+a_1(u+v,\xi,\lambda)a_6(v,\eta,\lambda)\\
&-a_6(u+v,\xi,\lambda)a_1(v,\eta,\lambda)-a_5(u,\xi,\eta)a_7(u+v,\xi,\lambda)a_7(v,\eta,\lambda)=0,\\
&a_6(u,\xi,\eta)a_5(v,\eta,\lambda)+a_1(u+v,\xi,\lambda)\\
&-a_1(u,\xi,\eta)a_1(v,\eta,\lambda)-a_7(u,\xi,\eta)a_4(u+v,\xi,\lambda)a_7(v,\eta,\lambda)=0,\\
&a_6(u,\xi,\eta)a_7(u+v,\xi,\lambda)a_1(v,\eta,\lambda)+a_5(u+v,\xi,\lambda)a_7(v,\eta,\lambda)\\
&-a_1(u,\xi,\eta)a_7(u+v,\xi,\lambda)a_5(v,\eta,\lambda)-a_7(u,\xi,\eta)a_6(u+v,\xi,\lambda)=0,\\
&a_7(u,\xi,\eta)a_1(u+v,\xi,\lambda)a_1(v,\eta,\lambda)+a_4(u,\xi,\eta)a_7(v,\eta,\lambda)\\
&-a_1(u,\xi,\eta)a_7(u+v,\xi,\lambda)-a_7(u,\xi,\eta)a_6(u+v,\xi,\lambda)a_6(v,\eta,\lambda)=0
\end{aligned}
\tag{1.10}
$$

plus six equations obtained by interchanging the sub-indices 1 and 4 as well as 5 and 6 in each of Equations (1.10). We call the six equations **counterparts** of (1.10).

Now we solve the equations obtained by letting $u=0$ and $\eta=\xi$ in (1.10) with respect to the variables $\{a_1(0,\xi,\xi),a_4(0,\xi,\xi),a_5(0,\xi,\xi),a_6(0,\xi,\xi),a_7(0,\xi,\xi)\}$. It is easy to prove that

Proposition 1.1 For a solution of equations (1.10), weight functions satisfy

$$
\begin{aligned}
&a_1(0,\xi,\xi)=a_4(0,\xi,\xi)=1,\\
&a_5(0,\xi,\xi)=a_6(0,\xi,\xi)=a_7(0,\xi,\xi)=a_8(0,\xi,\xi)=0.
\end{aligned}
\tag{1.11}
$$

Otherwise, up to the solution transformations A, B, C, D, and E, we have two trivial solutions of the colored Yang-Baxter equation (1.1). The first is

$$
\begin{aligned}
&a_1(u,\xi,\eta)=a_4(u,\xi,\eta)=a_5(u,\xi,\eta)=a_6(u,\xi,\eta)=H(u,\xi,\eta),\\
&a_2(u,\xi,\eta)=a_3(u,\xi,\eta)=a_7(u,\xi,\eta)=a_8(u,\xi,\eta)=1,
\end{aligned}
\tag{1.12a}
$$

where $H(u,\xi,\eta)$ is an arbitrary function of spectral parameter u and colored parameters ξ,η and the second

$$
\begin{aligned}
&a_1(u,\xi,\eta)=a_4(u,\xi,\eta)=a_5(u,\xi,\eta)=-a_6(u,\xi,\eta)=\frac{F(\xi)}{F(\eta)}exp(u),\\
&a_1(u,\xi,\eta)=a_2(u,\xi,\eta)=1,\qquad a_7(u,\xi,\eta)=a_8(u,\xi,\eta)=i,
\end{aligned}
\tag{1.12b}
$$

where $i^2 = -1$ and $F(\xi)$ is an arbitrary function of colored parameter ξ.

Definition 1.2 By a gauge solution of the colored Yang-Baxter equation (1.1) we mean the solution whose weight functions satisfy $a_2(u, \xi, \eta) = a_3(u, \xi, \eta) = 1$ and $a_7(u, \xi, \eta) = a_8(u, \xi, \eta)$ and the condition (1.11).

(1.11) is called an initial condition of gauge solutions. The condition is simple but very important. It will be quoted again and again in finding gauge solutions of the colored Yang-Baxter equation. For example, taking $v = -u$, $\lambda = \xi$ in (1.10) and their counterparts and using the initial condition (1.11), one has the following equations.

$$\begin{aligned}
&a_6(-u, \eta, \xi) = -a_6(u, \xi, \eta), \\
&a_5(-u, \eta, \xi) = -a_5(u, \xi, \eta), \\
&1 - a_1(u, \xi, \eta)a_1(-u, \eta, \xi) + a_6(u, \xi, \eta)a_5(-u, \eta, \xi) - a_7(u, \xi, \eta)a_7(-u, \eta, \xi) = 0, \\
&1 - a_4(u, \xi, \eta)a_4(-u, \eta, \xi) + a_5(u, \xi, \eta)a_6(-u, \eta, \xi) - a_7(u, \xi, \eta)a_7(-u, \eta, \xi) = 0.
\end{aligned} \tag{1.13}$$

The unitary condition of a solution of Yang-Baxter equation means

$$\check{R}(0, \xi, \xi) = E, \qquad \check{R}(u, \xi, \eta)\, \check{R}(-u, \eta, \xi) = g(u, \xi, \eta)\, E,$$

where E is the unit matrix and $g(u, \xi, \eta)$ a scalar function. Hence, it is easy to get from (1.13) that

Proposition 1.3: For gauge solutions $\check{R}(u, \xi, \eta)$ of the colored Yang-Baxter equation (1.1), the unitary condition is

$$\check{R}(u, \xi, \eta)\, \check{R}(-u, \eta, \xi) = (1 - a_5(u, \xi, \eta)\, a_6(u, \xi, \eta))\, E.$$

Differentiating both sides of all equations in (1.10) and their counterparts with respect to the variable v and letting $v = 0$, $\lambda = \eta$, by virtue of the initial condition (1.11) one

immediately obtains

$$
\begin{aligned}
&m_5(\eta) + a_5(u,\xi,\eta)\,a_1'(u,\xi,\eta) - a_1(u,\xi,\eta)\,a_5'(u,\xi,\eta) - m_6(\eta)\,a_7(u,\xi,\eta)^2 = 0,\\
&a_7'(u,\xi,\eta) + (m_1(\eta) - m_4(\eta))\,a_7(u,\xi,\eta) + m_7(\eta)\,(a_5(u,\xi,\eta)^2 - a_1(u,\xi,\eta)^2) = 0,\\
&a_6'(u,\xi,\eta) - m_6(\eta)\,a_1(u,\xi,\eta) + m_1(\eta)\,a_6(u,\xi,\eta) + m_7(\eta)\,a_5(u,\xi,\eta)\,a_7(u,\xi,\eta) = 0,\\
&a_1'(u,\xi,\eta) - m_1(\eta)\,a_1(u,\xi,\eta) + m_5(\eta)\,a_6(u,\xi,\eta) - m_7(\eta)\,a_4(u,\xi,\eta)\,a_7(u,\xi,\eta) = 0,\\
&a_6(u,\xi,\eta)\,a_7'(u,\xi,\eta) - a_7(u,\xi,\eta)\,a_6'(u,\xi,\eta)\\
&+m_1(\eta)\,a_6(u,\xi,\eta)\,a_7(u,\xi,\eta) - m_5(\eta)\,a_1(u,\xi,\eta)\,a_7(u,\xi,\eta) + m_7(\eta)\,a_5(u,\xi,\eta) = 0,\\
&a_7(u,\xi,\eta)\,a_1'(u,\xi,\eta) - a_1(u,\xi,\eta)\,a_7'(u,\xi,\eta)\\
&+m_1(\eta)\,a_1(u,\xi,\eta)\,a_7(u,\xi,\eta) - m_6(\eta)\,a_6(u,\xi,\eta)\,a_7(u,\xi,\eta) + m_7(\eta)\,a_4(u,\xi,\eta) = 0
\end{aligned}
\tag{1.14a}
$$

and their counterparts, where and throughout the paper we denote

$$
a_i'(u,\xi,\eta) = \frac{\partial}{\partial u} a_i(u,\xi,\eta), \quad m_i(\xi) = a_i'(u,\xi,\eta)|_{\{u=0,\eta=\xi\}},
$$

for $i = 1,4,5,6,7$.

We call $m_i(\xi)$ Hamiltonian coefficients of weight functions with respect to spectral parameter or simply coefficients. Sometimes we write m_i instead of $m_i(\xi)$ for brevity.

If we differentiate (1.10) with respect to u and let $u = 0$, $\eta = \xi$ and then replace the variables v and λ by u and η, we have

$$
\begin{aligned}
&m_6(\xi) + a_6(u,\xi,\eta)\,a_1'(u,\xi,\eta) - a_1(u,\xi,\eta)\,a_6'(u,\xi,\eta) - m_5(\xi)\,a_7(u,\xi,\eta)^2 = 0,\\
&a_7'(u,\xi,\eta) + (m_1(\xi) - m_4(\xi))\,a_7(u,\xi,\eta) + m_7(\xi)\,(a_6(u,\xi,\eta)^2 - a_1(u,\xi,\eta)^2) = 0,\\
&a_5'(u,\xi,\eta) - m_5(\xi)\,a_1(u,\xi,\eta) + m_1(\xi)\,a_5(u,\xi,\eta) + m_7(\xi)\,a_6(u,\xi,\eta)\,a_7(u,\xi,\eta) = 0,\\
&a_1'(u,\xi,\eta) - m_1(\xi)\,a_1(u,\xi,\eta) + m_6(\xi)\,a_5(u,\xi,\eta) - m_7(\xi)\,a_4(u,\xi,\eta)\,a_7(u,\xi,\eta) = 0,\\
&a_5(u,\xi,\eta)\,a_7'(u,\xi,\eta) - a_7(u,\xi,\eta)\,a_5'(u,\xi,\eta)\\
&+m_1(\xi)\,a_5(u,\xi,\eta)\,a_7(u,\xi,\eta) - m_6(\xi)\,a_1(u,\xi,\eta)\,a_7(u,\xi,\eta) + m_7(\xi)\,a_6(u,\xi,\eta) = 0,\\
&a_7(u,\xi,\eta)\,a_1'(u,\xi,\eta) - a_1(u,\xi,\eta)\,a_7'(u,\xi,\eta)\\
&+m_1(\xi)\,a_1(u,\xi,\eta)\,a_7(u,\xi,\eta) - m_5(\xi)\,a_5(u,\xi,\eta)\,a_7(u,\xi,\eta) + m_7(\xi)\,a_4(u,\xi,\eta) = 0
\end{aligned}
\tag{1.14b}
$$

and their counterparts.

Remark 1 : In this paper , the following trick will often be employed to obtain the equations like (1.14a) and (1.14b): We first do the calculation with respect to v and take $\lambda = \eta$ to yield some equations. Then repeat the same operation with u in place of v and take $\eta = \xi$ to yield another equation. Then we compare the two results to reduce to some of the formulas. For example, formula (1.8) is obtained by this method.

Comparing (1.14a) with (1.14b), we found that the trick, in fact, is to interchange the subindexes 5 and 6 (or 1 and 4) and then replace $m_i(\eta)$ by $m_i(\xi)$ in original equations (1.14a). We call this trick **symmetric operation** .

Remark 2: If the variable with respect to which we differentiate (1.10) is not a spectral parameter v but a colored parameter λ, the equations obtained are the same as (1.14a) if we still let $v = 0$ and $\lambda = \eta$. Of course, then the dot means the derivative with respect to the colored parameter η, that is the second colored variable in $a_i(u, \xi, \eta)$, and $m_i(\eta) = da_i(v, \eta, \lambda)/d\lambda|_{v=0,\lambda=\eta}$. Similarly, (1.14b) also represents the equations obtained by differentiating (1.10) with respect to the first colored variable ξ in $a_i(u, \xi, \eta)$, but in this case the dot means the derivative with respect to ξ and $m_i(\xi) = da_i(u, \xi, \eta)/d\lambda|_{u=0,\eta=\xi}$.

§2. The coefficients, curves and differential equations of weight functions

In this section we will discuss properties of Hamiltonian coefficients, curves and differential equations satisfied by weight functions.

It follows from the second equation of (1.14a) and its counterpart that

$$2a'_7(u, \xi, \eta) + m_7(\eta)\left(a_5(u, \xi, \eta)^2 + a_6(u, \xi, \eta)^2 - a_1(u, \xi, \eta)^2 - a_4(u, \xi, \eta)^2\right) = 0. \qquad (2.1)$$

If the symmetric operation is used then one also has

$$2a'_7(u, \xi, \eta) + m_7(\xi)\left(a_5(u, \xi, \eta)^2 + a_6(u, \xi, \eta)^2 - a_1(u, \xi, \eta)^2 - a_4(u, \xi, \eta)^2\right) = 0. \qquad (2.2)$$

We know $(u_5^2 + u_6^2 - u_1^2 - u_4^2) \not\equiv 0$ due to the initial condition (1.11). Compared the formulas (2.1) and (2.2), one shows

$$m_7(\xi) = m_7(\eta).$$

If $m_7(\xi) = 0$ then

$$a'_7(u, \xi, \eta) = 0, \qquad m_1(\eta) = m_4(\eta)$$

due to the second of (1.14a) and its counterpart. This implies $a_7(u, \xi, \eta)$ is a function of only the colored variables ξ and η.

Proposition 2.1: For gauge eight-vertex solutions, $m_7(\xi)$ is a constant independent of colored parameters and $a_7(u,\xi,\eta)$ is independent of spectral parameter if $m_7(\eta)=0$.

Remark 3: In fact, as mentioned in remark 2, the latter property of proposition 2.1 also holds for colored parameter, namely, $a_7(u,\xi,\eta)$ will be independent of the colored parameter ξ (or η) if $d/d\xi(a_7(u,\xi,\eta))|_{u=0,\eta=\xi}=0$ (or $d/d\eta(a_7(u,\xi,\eta))|_{u=0,\xi=\eta}=0$).

In what follows we denote $m_7(\xi)=\alpha$.

Furthermore, if $\alpha=0$, letting $u=0$, $\eta=\xi$ in the following equation

$$a_1'(u,\xi,\eta)=m_6(\eta)\,a_6(u,\xi,\eta)-m_1(\eta)\,a_1(u,\xi,\eta), \tag{2.3}$$

which is from the sixth in (1.14a), we obtain $m_1(\eta)=0$ owing to the initial condition (1.11). Therefore $m_4(\xi)=0$. Similarly, we have

$$a_6'(u,\xi,\eta)=m_1(\eta)\,a_6(u,\xi,\eta)-m_5(\eta)\,a_1(u,\xi,\eta) \tag{2.4}$$

from the fifth in (1.14a) and then

$$m_6(\eta)=-m_5(\eta). \tag{2.5}$$

Substituting $m_6(\eta)=-m_5(\eta)$ and $m_1(\eta)=m_4(\eta)=0$ into the third and fourth of (1.14a) and their counterparts, we have the following differential equations.

$$\begin{aligned} a_1'(u,\xi,\eta)&=-m_5(\eta)a_6(u,\xi,\eta), & a_4'(u,\xi,\eta)&=m_5(\eta)a_5(u,\xi,\eta),\\ a_6'(u,\xi,\eta)&=-m_5(\eta)a_1(u,\xi,\eta), & a_5'(u,\xi,\eta)&=m_5(\eta)a_4(u,\xi,\eta). \end{aligned} \tag{2.6}$$

Therefore, if $\alpha=0$ then weight functions satisfy

$$\frac{d^2}{du^2}a_i(u,\xi,\eta)=m_5(\eta)^2\,a_i(u,\xi,\eta), \qquad i=1,4,5,6. \tag{2.7}$$

Furthermore, using the symmetric operation

$$\frac{d^2}{du^2}a_i(u,\xi,\eta)=m_5(\xi)^2\,a_i(u,\xi,\eta), \qquad i=1,4,5,6 \tag{2.8}$$

hold. Hence $m_5(\eta)$ actually is a constant independent of colored parameters and not identically zero, otherwise, the solutions will be independent of spectral parameter. Thus we can let $m_5(\eta)=\beta$.

The argument above implies the following proposition.

Proposition 2.2 For a gauge eight-vertex solution, there exists at least one between $m_7(\xi)$ and $m_5(\xi)$ (or $m_6(\xi)$) which is not zero identically. Otherwise, the solution will be independent of spectral parameter.

As for the Hamiltonian coefficients $m_1(\xi)$ and $m_4(\xi)$ we have

Proposition 2.3*: For gauge eight-vertex solutions of the colored Yang-Baxter equation (1.1)

$$m_1(\xi)^2 - m_4(\xi)^2 = 0.$$

Proof: As first step, we regard the weight functions w_i $(i = 1, 4, 5, 6, 7)$ in (1.10) and their counterparts as indeterminates, The left side of each of (1.10) and its counterpart are polynomial functions of the indeterminates. After eliminating the five indeterminates $\{w_1, w_4, w_5, w_6, w_7\}$ in numerically increasing order with respect to the system of equations (1.10), we can obtain seven equations which do not contain the indeterminates w_i $(i = 1, 4, 5, 6, 7)$. Then differentiating them with respect to the spectral variable v , letting $v = 0, \lambda = \eta$ and then substituting the initial values (1.11) into the resulting ones, we obtain the following seven polynomial equations

$$\begin{aligned}
&m_7u_1^3 - m_7u_1u_5^2 - 3\,m_1u_1u_7 + m_4u_1u_7 - m_7u_4u_7^2 - m_7u_4 + m_5u_6u_7 + m_6u_6u_7 = 0,\\
&-m_6u_1u_4 + m_7u_1u_6u_7 + m_7u_4u_5u_7 + m_1u_4u_6 + m_4u_4u_6 - m_6u_5u_6 - m_5u_7^2 + m_6 = 0,\\
&m_7u_1^2 - m_7u_4^2 - m_7u_5^2 + m_7u_6^2 + 2\,m_4u_7 - 2\,m_1u_7 = 0,\\
&-m_5u_1u_4 + m_1u_1u_5 + m_4u_1u_5 + m_7u_1u_6u_7 + m_7u_4u_5u_7 - m_5u_5u_6 - m_6u_7^2 + m_5 = 0,\\
&m_7u_1^2u_6 - m_6u_1u_7 - m_5u_1u_7 - m_7u_5^2u_6 + m_7u_5u_7^2 + m_7u_5 + m_4u_6u_7 + m_1u_6u_7 = 0,\\
&m_7u_1^3u_5 - m_6u_1u_4u_7 - m_7u_1u_5^3 + 2\,m_4u_1u_5u_7 - 2\,m_1u_1u_5u_7 + m_7u_1u_6\\
&-m_7u_4u_5u_7^2 + m_5u_5u_6u_7 + m_6u_7^3 - m_5u_7 = 0,\\
&-m_7u_1^2u_4 + m_7u_1u_7^2 + m_7u_1 + m_7u_4u_5^2 + m_4u_4u_7 + m_1u_4u_7 - m_5u_5u_7 - m_6u_5u_7 = 0.
\end{aligned} \tag{2.9}$$

As second step, we think of m_i as indeterminates and first eliminate m_1, m_4 and m_5 to get two systems of equations, which are equivalent to (2.9). The first is

$$\begin{aligned}
&-m_5u_1u_4 + m_1u_1u_5 + m_4u_1u_5 + m_7u_1u_6u_7 + m_7u_4u_5u_7 - m_5u_5u_6 - m_6u_7^2 + m_5 = 0,\\
&-m_7u_1^3u_5 + m_7u_1u_4^2u_5 + 2\,m_5u_1u_4u_7 + m_7u_1u_5^3 - m_7u_1u_5u_6^2 - 4\,m_4u_1u_5u_7\\
&-2\,m_7u_1u_6u_7^2 - 2\,m_7u_4u_5u_7^2 + 2\,m_5u_5u_6u_7 + 2\,m_6u_7^3 - 2\,m_5u_7 = 0,\\
&-m_7u_1u_4^2u_5 + m_6u_1u_4u_7 + m_7u_1u_5u_6^2 - m_7u_1u_6 + m_7u_4u_5u_7^2 - m_5u_5u_6u_7\\
&-m_6u_7^3 + m_5u_7 = 0.
\end{aligned} \tag{2.10}$$

which contains m_1, m_4 and m_5. The second is

$$\begin{aligned}
&(u_1u_4 + u_5u_6 - u_7^2 - 1)\,(-m_7u_1u_4^2u_5 + m_6u_1u_4u_7 + m_7u_1u_5u_6^2 \\
&-m_7u_1u_6 + m_7u_4u_5 - m_6u_5u_6u_7) = 0, \\
&u_1\,(u_1u_4 + u_5u_6 - u_7^2 - 1)\,(m_7u_1u_4u_5^2 - m_6u_1u_5u_7 - m_7u_4^2u_5u_6 + m_6u_4u_6u_7 \\
&-m_7u_5^3u_6 + m_7u_5^2 + m_7u_5u_6^3 - m_7u_6^2) = 0, \\
&u_1\,(u_1u_4 + u_5u_6 - u_7^2 - 1)\,(-m_7u_4^3u_5u_6 + m_6u_4^2u_6u_7 + m_7u_4u_5^2u_7^2 + m_7u_4u_5u_6^3 \\
&-m_7u_4u_6^2 - m_6u_5^2u_6u_7 - m_6u_5u_7^3 + m_6u_5u_7) = 0, \\
&u_1\,(u_1u_4 + u_5u_6 - u_7^2 - 1)\,(-m_7u_1u_5^2u_6 + m_7u_1u_5 - m_7u_4^3u_5 + m_6u_4^2u_7 \\
&+m_7u_4u_5^3 + m_7u_4u_5u_6^2 - m_7u_4u_6 - m_6u_5^2u_7) = 0,
\end{aligned} \tag{2.11}$$

which do not contain m_1, m_4 and m_5. So the free fermion condition[17]

$$u_1u_4 + u_5u_6 - 1 - u_7^2 = 0 \tag{2.12}$$

or

$$\begin{aligned}
&-m_7u_1u_4^2u_5 + m_6u_1u_4u_7 + m_7u_1u_5u_6^2 - m_7u_1u_6 + m_7u_4u_5 - m_6u_5u_6u_7 = 0, \\
&m_7u_1u_4u_5^2 - m_6u_1u_5u_7 - m_7u_4^2u_5u_6 + m_6u_4u_6u_7 - m_7u_5^3u_6 + m_7u_5^2 \\
&+m_7u_5u_6^3 - m_7u_6^2 = 0, \\
&-m_7u_4^3u_5u_6 + m_6u_4^2u_6u_7 + m_7u_4u_5^2u_7^2 + m_7u_4u_5u_6^3 - m_7u_4u_6^2 - m_6u_5^2u_6u_7 \\
&-m_6u_5u_7^3 + m_6u_5u_7 = 0, \\
&-m_7u_1u_5^2u_6 + m_7u_1u_5 - m_7u_4^3u_5 + m_6u_4^2u_7 + m_7u_4u_5^3 + m_7u_4u_5u_6^2 - m_7u_4u_6 \\
&-m_6u_5^2u_7 = 0
\end{aligned} \tag{2.13}$$

will hold. In the third step, applying the fourth in (2.13) as a main equation to kill the indeterminate m_6 in the three ones remained in (2.13) and then doing factorization of the new polynomial equations after killing m_6, one can obtain that

$$\begin{aligned}
&-m_7u_1u_5^2u_6 + m_7u_1u_5 - m_7u_4^3u_5 + m_6u_4^2u_7 + m_7u_4u_5^3 + m_7u_4u_5u_6^2 - m_7u_4u_6 \\
&-m_6u_5^2u_7 = 0, \\
&m_7\,(u_5u_6 - 1)\,(u_1^2u_5 - 2\,u_1u_4u_6 + u_4^2u_5 - u_5^3 + u_5u_6^2)\,u_5 = 0, \\
&m_7\,(u_5u_6 - 1)\cdot \\
&(-u_1u_4^2u_6 + u_1u_5^2u_6 + u_1u_5u_7^2 - u_1u_5 + u_4^3u_5 - u_4u_5^3 - u_4u_6u_7^2 + u_4u_6)\,u_5 = 0, \\
&m_7\,(u_5u_6 - 1)\,(-u_1^2u_4 + 2\,u_1u_5u_6 + u_4^3 - u_4u_5^2 - u_4u_6^2)\,u_5 = 0
\end{aligned} \tag{2.14}$$

is equivalent to (2.13). It follows from the first of (2.14) that $m_7 \not\equiv 0$. Otherwise, thanks to proposition 2.2 $a_4(u,\xi,\eta)^2 = a_5(u,\xi,\eta)^2$. The latter is impossible thanks to the initial condition (1.11). Hence the following three equations

$$\begin{aligned}
&u_1^2 u_5 - 2\,u_1 u_4 u_6 + u_4^2 u_5 - u_5^3 + u_5 u_6^2 = 0,\\
&-u_1 u_4^2 u_6 + u_1 u_5^2 u_6 + u_1 u_5 u_7^2 - u_1 u_5 + u_4^3 u_5 - u_4 u_5^3 - u_4 u_6 u_7^2 + u_4 u_6 = 0, \qquad (2.15)\\
&-u_1^2 u_4 + 2\,u_1 u_5 u_6 + u_4^3 - u_4 u_5^2 - u_4 u_6^2 = 0
\end{aligned}$$

and the first of (2.14) is equivalent to (2.14), where we use the initial condition (1.11) again to yield $1 - u_5 u_6 \not\equiv 0$.

Finally, if we differentiate (2.12) and the third equation of (2.15) with respect to u and let $u = 0$, $\xi = \eta$ and then apply the initial conditions (1.11) again , we come to the conclusion of proposition 2.3.

Remark 4: When we do the operation of eliminating indeterminates with respect to a system of polynomial equations, according to the theorem of zero structure of algebraic varieties [22], the coefficient of the term with the highest degree of the indeterminate in main polynomial equation (to be eliminated in other polynomials) should not be identified with zero. In the event it is identified with zero, we should add the coefficient into the system of equations to produce a new system of equations. Otherwise, it is possible to lose some solutions. For example, when we use the fourth equation in (2.13) to eliminate m_6 in the three remaining ones in (2.13), because the coefficient of m_6 in the fourth one of (2.13) is $u_7(u_4^2 - u_5^2)$ which does not identify with zero due to the initial condition (1.11), (2.14) is equivalent to the system of equations (2.13).

From the argument of proving proposition 2.3 above we see the system of equations (2.9) is equivalent to two groups of equations. The first is (2.10), (2.15) plus the first of (2.14). The second is (2.10) and (2.12), the free fermion condition.

Now we consider the two cases respectively. For the first case we differentiate the second equation in (2.15) and take $u = 0$, $\xi = \eta$. Then we substitute the initial condition (1.11) into the result to get

$$m_5(\eta) = m_6(\eta). \qquad (2.16)$$

By doing factorization of the equation obtained by eliminating u_4 in the third equation of (2.15) and by using the second equation of (2.15) , we can get

$$2\,u_6\,(u_6 - u_5)\,(u_6 + u_5)\,(u_1 - u_5)\,(u_1 + u_5)\,(u_1 - u_6)\,(u_1 + u_6) = 0. \qquad (2.17)$$

Together, (2.17) and the initial condition (1.11) imply

$$a_5(u,\xi,\eta) = a_6(u,\xi,\eta) \qquad (2.18a)$$

or

$$a_5(u,\xi,\eta) = -a_6(u,\xi,\eta) \tag{2.18b}$$

hold. Substituting (2.18a) into the third equation in (2.15), we then see

$$a_1(u,\xi,\eta) = a_4(u,\xi,\eta). \tag{2.19}$$

Substituting (2.18a) and (2.19) into the first equation in (2.14), we have

$$(u_5 - u_1(u_5+u_1)(\alpha\, u_5 u_1 - m_6\, u_7) = 0. \tag{2.20}$$

So

$$\alpha\, a_1(u,\xi,\eta)\, a_5(u,\xi,\eta) - m_6(\eta)\, a_7(u,\xi,\eta) = 0, \tag{2.21}$$

here the initial condition (1.11) is used again. (2.21) implies $m_6(\eta) \not\equiv 0$, or α will also identify with zero. But this will contradict proposition 2.2. If we do the symmetric operation with respect to (2.21) and use (2.19), then

$$\alpha\, a_1(u,\xi,\eta)\, a_5(u,\xi,\eta) - m_6(\xi)\, a_7(u,\xi,\eta) = 0, \tag{2.22}$$

(2.21) and (2.22) will imply $m_6(\eta)$ is also a constant independent of colored parameter. We let $m_5(\eta) = \beta$.

If $a_5(u,\xi,\eta) = -a_6(u,\xi,\eta)$ we should have $m_6(\eta) = 0$ and (2.21) still holds. Then $\alpha = 0$. It is clearly impossible thanks to proposition 2.2.

Combining (2.18a), (2.19), (2.21) with the third equation in (1.14a) we obtain

$$(a_5'(u,\xi,\eta))^2 = \beta^2 - (\beta^2 - m_1(\eta)^2 + \alpha^2)\, a_5(u,\xi,\eta)^2 + \alpha^2\, a_5(u,\xi,\eta)^4. \tag{2.23}$$

Using the symmetric operation we can show $m_1(\eta)$ is also a constant independent of colored parameter. Let $m_1(\eta) = \gamma$. Similarly,

$$(a_1'(u,\xi,\eta))^2 = \beta^2 - (\beta^2 - \gamma^2 + \alpha^2)\, a_1(u,\xi,\eta)^2 + \alpha^2\, a_1(u,\xi,\eta)^4 \tag{2.24}$$

holds if we combine (2.18a), (2.19), (2.21) with the fourth in (1.14a). Substituting (2.16), (2.18a), (2.19) and (2.21) into the second of (2.10) we fund that the algebraic curve satisfied by the weight functions $a_1(u,\xi,\eta)$ and $a_5(u,\xi,\eta)$ is

$$\alpha^2\, u_1^2 u_5^2 - \beta^2\, u_5^2 - \beta^2 u_1^2 + 2\beta\,\gamma\, u_1 u_5 + \beta^2 = 0. \tag{2.25}$$

From the second group of equations, i.e. (2.10) and (2.12), it is easy to obtain

$$\begin{aligned}
&1 + a_7(u,\xi,\eta)^2 - a_1(u,\xi,\eta)a_4(u,\xi,\eta) - a_5(u,\xi,\eta)a_6(u,\xi,\eta) = 0,\\
&\alpha\,(a_1(u,\xi,\eta)a_6(u,\xi,\eta) + a_4(u,\xi,\eta)a_5(u,\xi,\eta)) = (m_5(\eta) + m_6(\eta))\, a_7(u,\xi,\eta),\\
&\alpha\,(a_1(u,\xi,\eta)^2 + a_6(u,\xi,\eta)^2 - a_4(u,\xi,\eta)^2 - a_5(u,\xi,\eta)^2) = 4m_1(\eta)\, a_7(u,\xi,\eta).
\end{aligned} \tag{2.26}$$

Then $m_1(\eta) + m_4(\eta) = 0$. If we do the symmetric operation with respect to the third equation of (2.26) then we also obtain

$$\alpha\,(a_1(u,\xi,\eta)^2 + a_5(u,\xi,\eta)^2 - a_4(u,\xi,\eta)^2 - a_6(u,\xi,\eta)^2) = 4m_1(\xi)\,a_7(u,\xi,\eta). \tag{2.27}$$

Letting $\eta = \xi$ in the third equation in (2.26) and (2.27) we reduce to

$$a_6(u,\xi,\xi)^2 = a_5(u,\xi,\xi)^2. \tag{2.28}$$

Therefore, (2.16) and (2.28) give the following proposition.

Proposition 2.4: For gauge eight-vertex solutions of the colored Yang-Baxter equation (1.1) the Hamiltonian coefficients $m_5(\eta)$ and $m_6(\eta)$ satisfy

$$m_5(\eta)^2 - m_6(\eta)^2 = 0.$$

From (2.26) and the second equation in (1.14a), we can calculate the weight function $a_7(u,\xi,\eta)$ obeys

$$\begin{aligned} &a_7'(u,\xi,\eta)^2 \\ &= \alpha^2 - ((m_5(\eta) + m_6(\eta))^2 - 4\,m_1^2(\eta) - 2\,\alpha^2)a_7(u,\xi,\eta)^2 + \alpha^2 a_7(u,\xi,\eta)^4. \end{aligned} \tag{2.29}$$

Furthermore, as we said in remark 1, we can show $(m_5(\eta)^2 + m_6\eta))^2 - 2m_1(\eta)^2$ is a constant independent of colored parameter using the symmetric operation. Let $\delta^2 = (m_5(\eta)^2 + m_6(\eta))^2 - 2m_1(\eta)^2$.

We conclude this section by the following theorem.

THEOREM 2.5*: For a gauge eight-vertex type solution, its weight functions must satisfy one of two systems of equations. The first is composed of (2.18a), (2.19), (2.21), (2.23), (2.24) and (2.25). The second is composed of (2.26) and (2.29).

§3. Gauge eight-vertex type solutions

In this section $sn(\zeta)$ and $cd(\zeta) = cn(\zeta)/dn(\zeta)$ are Jacobian elliptic functions.

Now we describe how to write down all gauge solutions of eight-vertex type of the colored Yang-Baxter (1.1) and classify them into two types called Baxter type and Free-Fermion type.

3.1 Baxter type solutions.

We consider the first case in theorem 2.6. Since the curve (2.25) only includes two weight functions $a_1(u,\xi,\eta)$ and $a_5(u,\xi,\eta)$, we can parameterize $a_1(u,\xi,\eta)$ and $a_5(u,\xi,\eta)$

as one-parameter functions. If $\beta \pm \alpha \pm \gamma \neq 0$, (2.23) and (2.24) are elliptic differential equations. Therefore, the solutions should be

$$
\begin{aligned}
a_1(u,\xi,\eta) &= a_4(u,\xi,\eta) = \frac{sn(\lambda\, u + F(\xi) - F(\eta) + \mu)}{sn(\mu)}, \\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1, \\
a_5(u,\xi,\eta) &= a_6(u,\xi,\eta) = \pm\frac{sn(\lambda\, u + F(\xi) - F(\eta))}{sn(\mu)}, \\
a_7(u,\xi,\eta) &= \pm k\, sn(\lambda\, u + F(\xi) - F(\eta))\, sn(\lambda\, u + F(\xi) - F(\eta) + \mu)
\end{aligned}
\tag{3.1}
$$

where k, as the modules of Jacobi elliptic function, is an arbitrary constant.

If $\beta \pm \alpha \pm \gamma = 0$, the elliptic solutions (3.1) will degenerate into trigonometric solutions

$$
\begin{aligned}
a_1(u,\xi,\eta) &= a_4(u,\xi,\eta) = \frac{tan(\lambda\, u + F(\xi) - F(\eta) + \mu)}{tan(\mu)}, \\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1, \\
a_5(u,\xi,\eta) &= a_6(u,\xi,\eta) = \pm\frac{tan(\lambda\, u + F(\xi) - F(\eta))}{tan(\mu)}, \\
a_7(u,\xi,\eta) &= \pm tan(\lambda\, u + F(\xi) - F(\eta))\, tan(\lambda\, u + F(\xi) - F(\eta) + \mu).
\end{aligned}
\tag{3.2}
$$

In (3.1) and (3.2) $\lambda \neq 0$, $\mu \neq 0$ are two arbitrary constants and $F(\xi)$ an arbitrary function.

3.2 Free-Fermion type solutions.

Now we consider the second case in theorem (2.6). According to proposition 2.4 it is divided into two sub-cases, $m_5(\xi) = m_6(\xi)$ and $m_5(\xi) = -m_6(\xi)$.

(3.2a) For the sub-case of $m_5(\xi) = m_6(\xi)$

Let $m_5(\xi) \not\equiv 0$ (we will put the case of $m_5(\eta) = 0$ into the second sub-case). It is clear that $\alpha \neq 0$ due to the second of (2.26). For brevity we let $\alpha = 1$ up to the solution transformation D. When

$$m_5(\xi)^2 - m_1(\xi)^2 \not\equiv 0$$

and

$$m_5(\xi)^2 - m_1(\xi)^2 \not\equiv 1$$

the equation (2.29) then is an elliptic differential equation and, from the remarks 2, 3 and the initial condition (1.11), should have solutions

$$a_7(u,\xi,\eta) = k\, sn(\lambda\, u + F(\xi) - F(\eta))\, cd(\lambda\, u + F(\xi) - F(\eta)), \tag{3.3}$$

where k, as the module of elliptic function, and λ are two arbitrary constants and $F(\xi)$ an arbitrary function with the constriction $k\,\lambda = 1$.

Substituting (3.3) into the second of (1.14a) and its counterpart as well as the first and second of (2.26) and using elliptic function identities we have

$$cd^2 - sn^2 + 2\,m_1(\eta)k\;cd\;sn + u_5^2 - u_1^2 = 0, \tag{3.4a}$$

$$cd^2 - sn^2 - 2\,m_1(\eta)k\;cd\;sn + u_6^2 - u_4^2 = 0, \tag{3.4b}$$

$$u_1u_4 + u_5u_6 - cd^2 - sn^2 = 0, \tag{3.4c}$$

$$u_1u_6 + u_4u_5 - 2\,m_5(\eta)k\;cd\;sn = 0, \tag{3.4d}$$

where $m_5(\eta)$ and $m_1(\eta)$ are arbitrary functions satisfying

$$m_5(\eta)^2 - m_1(\eta)^2 = \frac{1}{k^2}.$$

In the formulas (3.4)s and in what follows we simply write sn, cd instead of elliptic functions $sn(\lambda\,u + F(\xi) - F(\eta))$ and $cd(\lambda\,u + F(\xi) - F(\eta))$ for brevity. Using the symmetric operation we also have

$$cd^2 - sn^2 + 2\,m_4(\xi)k\;cd\;sn + u_5^2 - u_4^2 = 0.$$

Since $m_1(\xi) + m_4(\xi) = 0$ one obtains

$$cd^2 - sn^2 - 2\,m_1(\xi)k\;cd\;sn + u_5^2 - u_4^2 = 0. \tag{3.4e}$$

From (3.4c), (3.4d) and (3.4a) one also obtains

$$-(cd^2 + sn^2)u_1 + (cd^2 - sn^2 + 2\,m_1(\eta)k\;sn\;cd)u_4 + 2\,m_5(\eta)k\;sn\;cd\;u_5 = 0. \tag{3.5a}$$

If we do the symmetric operation with respect to the counterpart of (3.5a) then

$$-(cd^2 + sn^2)u_4 + (cd^2 - sn^2 - 2\,m_1(\xi)k\;sn\;cd)u_1 + 2\,m_5(\xi)k\;sn\;cd\;u_5 = 0. \tag{3.5b}$$

Similarly, one has

$$(sn^2 + 2\,m_1(\eta)k\;sn\;cd - cd^2)u_5 - (cd^2 + sn^2)u_6 + 2\,m_5(\eta)k\;sn\;cd\;u_4 = 0, \tag{3.6a}$$

$$(sn^2 + 2\,m_1(\xi)k\;cd\;sn - cd^2)u_6 + (cd^2 + sn^2)u_5 + 2\,m_5(\xi)k\;cd\;sn\;u_4 = 0. \tag{3.6b}$$

Solving the equations (3.5a), (3.5b), (3.6a) and (3.6b) with respect to $\{u_1, u_4, u_5, u_6\}$ we have

$$\frac{a_4(u,\xi,\eta)}{a_1(u,\xi,\eta)} = \frac{H_4}{H_1}, \qquad \frac{a_6(u,\xi,\eta)}{a_5(u,\xi,\eta)} = \frac{H_6}{H_5},$$

where

$$\begin{aligned}
H_1 &= (m_5(\xi)+m_5(\eta))\,cd + (m_1(\xi)m_5(\eta)+m_5(\xi)m_1(\eta))\,k\,sn,\\
H_4 &= (m_5(\xi)+m_5(\eta))\,cd - (m_1(\xi)m_5(\eta)+m_5(\xi)m_1(\eta))\,k\,sn,\\
H_5 &= (m_5(\xi)+m_5(\eta))\,sn + (m_1(\xi)m_5(\eta)-m_5(\xi)m_1(\eta))\,k\,cd,\\
H_6 &= (m_5(\xi)+m_5(\eta))\,sn - (m_1(\xi)m_5(\eta)-m_5(\xi)m_1(\eta))\,k\,cd.
\end{aligned} \tag{3.6}$$

Let $u_1 = H_1 X$, $u_4 = H_4 X$ and $u_5 = H_5 Y$, $u_6 = H_6 Y$. From (3.4a) and (3.4e) one obtains

$$\begin{aligned}
(H_1^2 - H_4^2)X^2 &= 4\ (m_5(\xi)+m_5(\eta))\,(m_5(\eta)m_1(\xi)+m_5(\xi)m_1(\eta))\ k\,sn\,cd\,X^2\\
&= 2(m_1(\xi)+m_1(\eta)k\,sn\,cd,\\
(H_5^2 - H_6^2)Y^2 &= 4\ (m_5(\xi)+m_5(\eta))\,(m_5(\eta)m_1(\xi)-m_5(\xi)m_1(\eta))\,k\,sn\,cd\,Y^2\\
&= 2(m_1(\xi)-m_1(\eta))k\,sn\,cd.
\end{aligned} \tag{3.7}$$

It is easy to show using $m_5(\xi)^2 - m_1(\xi)^2 = 1/k^2$ that

$$\begin{aligned}
X^2 &= \frac{m_1(\xi)+m_1(\eta)}{2(m_5(\xi)+m_5(\eta))(m_1(\xi)m_5(\eta)+m_5(x)m_1(\eta))}\\
&= \frac{1+k^2(m_5(\xi)m_5(\eta)-m_1(\xi)m_1(\eta))}{2(m_5(\xi)+m_5(\eta))^2} = \frac{-1+k^2(m_5(\xi)m_5(\eta)+m_1(\xi)m_1(\eta))}{2(m_1(\xi)m_5(\eta)+m_5(\xi)m_1(\eta))^2\,k^2},\\
Y^2 &= \frac{m_1(\xi)-m_1(\eta)}{2(m_5(\xi)+m_5(\eta))(m_1(\xi)m_5(\eta)-m_5(x)m_1(\eta))}\\
&= \frac{1+k^2(m_5(\xi)m_5(\eta)+m_1(\xi)m_1(\eta))}{2(m_5(\xi)+m_5(\eta))^2} = \frac{-1+k^2(m_5(\xi)m_5(\eta)-m_1(\xi)m_1(\eta))}{2(m_1(\xi)m_5(\eta)-m_5(\xi)m_1(\eta))^2\,k^2}.
\end{aligned}$$

Hence gauge solutions of the colored Yang-Baxter equation (1.1) should obey the following forms

$$\begin{aligned}
a_1(u,\xi,\eta) &= A(\xi,\eta)\,cd(\lambda\,u+F(\xi)-F(\eta)) + B(\xi,\eta)\,sn(\lambda\,u+F(\xi)-F(\eta)),\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1,\\
a_4(u,\xi,\eta) &= A(\xi,\eta)\,cd(\lambda\,u+F(\xi)-F(\eta)) - B(\xi,\eta)\,sn(\lambda\,u+F(\xi)-F(\eta)),\\
a_5(u,\xi,\eta) &= C(\xi,\eta)\,sn(\lambda\,u+F(\xi)-F(\eta)) + D(\xi,\eta)\,cd(\lambda\,u+F(\xi)-F(\eta)),\\
a_6(u,\xi,\eta) &= C(\xi,\eta)\,sn(\lambda\,u+F(\xi)-F(\eta)) - D(\xi,\eta)\,cd(\lambda\,u+F(\xi)-F(\eta)),\\
a_7(u,\xi,\eta) &= \pm k\,sn(\lambda\,u+F(\xi)-F(\eta))\,cd(\lambda\,u+F(\xi)-F(\eta)),
\end{aligned} \tag{3.8}$$

where k, as the module of the elliptic functions, is an arbitrary constant and

$$
\begin{aligned}
A(\xi,\eta) &= \sqrt{(1+G(\xi)G(\eta)-H(\xi)H(\eta))/2},\\
B(\xi,\eta) &= \sqrt{(-1+G(\xi)G(\eta)+H(\xi)H(\eta))/2},\\
C(\xi,\eta) &= \delta\sqrt{(1+G(\xi)G(\eta)+H(\xi)H(\eta))/2},\\
D(\xi,\eta) &= \delta\sqrt{(-1+G(\xi)G(\eta)-H(\xi)H(\eta))/2}\,M,
\end{aligned}
\tag{3.9}
$$

where $\delta^2 = 1$ and

$$
M = \frac{H(\xi)G(\eta)-G(\xi)H(\eta)}{\sqrt{(H(\xi)G(\eta)-G(\xi)H(\eta))^2}}
$$

and $G(\xi),\ H(\xi)$ satisfy $G(\xi)^2 - H(\xi)^2 = 1$. If we consider the solution transformation D then the restrictive condition $k\,\lambda = 1$ can be cancelled, namely, $\lambda \neq 0$ is also an arbitrary constant.

When the module $k = 1$ the Jacobian elliptic functions cd and sn should degenerate into 1 and $tanh$. Hence, we have

$$
\begin{aligned}
a_1(u,\xi,\eta) &= A(\xi,\eta) + B(\xi,\eta)\,tanh(\lambda\,u + F(\xi) - F(\eta)),\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1,\\
a_4(u,\xi,\eta) &= A(\xi,\eta) - B(\xi,\eta)\,tanh(\lambda\,u + F(\xi) - F(\eta)),\\
a_5(u,\xi,\eta) &= C(\xi,\eta)\,tanh(\lambda\,u + F(\xi) - F(\eta)) + D(\xi,\eta),\\
a_6(u,\xi,\eta) &= C(\xi,\eta)\,tanh(\lambda\,u + F(\xi) - F(\eta)) - D(\xi,\eta),\\
a_7(u,\xi,\eta) &= \pm tanh(\lambda\,u + F(\xi) - F(\eta)),
\end{aligned}
\tag{3.10}
$$

where $A(\xi,\eta)$, $B(\xi,\eta)$, $C(\xi,\eta)$ and $D(\xi,\eta)$ are defined by (3.9) and $\lambda \neq 0$ is an arbitrary constant.

If $m_5(\eta)^2 - m_1(\eta)^2 = 0$ then the differential equation (2.29) can be rewritten as

$$
a_7'(u,\xi,\eta)^2 = \alpha^2(1 + 2\,a_7(u,\xi,\eta)^2 + a_7(u,\xi,\eta)^4).
$$

So, following the calculation of (3.8) we can show that the gauge solutions are

$$\begin{aligned}
a_1(u,\xi,\eta) &= X\left(\frac{G(\xi)+G(\eta)}{cos(\lambda\, u+F(\xi)-F(\eta))}+2G(\xi)G(\eta)sin(\lambda\, u+F(\xi)-F(\eta))\right),\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta)=1,\\
a_4(u,\xi,\eta) &= X\left(\frac{G(\xi)+G(\eta)}{cos(\lambda\, u+F(\xi)-F(\eta))}-2G(\xi)G(\eta)sin(\lambda\, u+F(\xi)-F(\eta))\right),\\
a_5(u,\xi,\eta) &= Y\left(\frac{G(\xi)-G(\eta)}{cos(\lambda\, u+F(\xi)-F(\eta))}+2G(\xi)G(\eta)sin(\lambda\, u+F(\xi)-F(\eta))\right),\\
a_6(u,\xi,\eta) &= Y\left(-\frac{G(\xi)-G(\eta)}{cos(\lambda\, u+F(\xi)-F(\eta))}+2G(\xi)G(\eta)sin(\lambda\, u+F(\xi)-F(\eta))\right),\\
a_7(u,\xi,\eta) &= \pm tan(\lambda\, u+F(\xi)-F(\eta)),
\end{aligned} \tag{3.11}$$

where

$$X=\frac{1}{2\sqrt{G(\xi)G(\eta)}},\qquad Y=\pm X \tag{3.12}$$

and $G(\xi)$ is an arbitrary function.

(3.2b) For the sub-case of $m_5(\xi)=-m_6(\xi)$

In the case of $m_5(\eta)=-m_6(\eta)$ weight functions of gauge solutions are

$$\begin{aligned}
a_1(u,\xi,\eta) &= a_4(u,\xi,\eta)=\frac{cosh(\lambda\, u+F(\xi)-F(\eta))}{cos(\mu\, u+G(\xi)-G(\eta))},\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta)=1,\\
a_5(u,\xi,\eta) &= -a_6(u,\xi,\eta)=\pm\frac{sinh(\lambda\, u+F(\xi)-F(\eta))}{cos(\mu\, u+G(\xi)-G(\eta))},\\
a_7(u,\xi,\eta) &= \pm tan(\mu\, u+G(\xi)-G(\eta)),
\end{aligned} \tag{3.13}$$

where λ and μ are two arbitrary constants, but not zero simultaneously, and $F(\xi)$, $G(\xi)$ are two arbitrary functions.

To prove it we first consider the sub-case of $\alpha\neq 0$. Then it follows that

$$a_1(u,\xi,\eta)a_6(u,\xi,\eta)+a_4(u,\xi,\eta)a_5(u,\xi,\eta)=0$$

by (2.26) and

$$a_1(u,\xi,\eta)a_5(u,\xi,\eta)+a_4(u,\xi,\eta)a_6(u,\xi,\eta)=0$$

by the symmetric operation. Since $a_1(u,\xi,\eta)\neq -a_4(u,\xi,\eta)$ owing to the initial condition

(1.11). So one can get

$$
\begin{aligned}
&a_1(u,\xi,\eta) = a_4(u,\xi,\eta),\\
&a_5(u,\xi,\eta) = -a_6(u,\xi,\eta),\\
&a_1(u,\xi,\eta)^2 - a_5(u,\xi,\eta)^2 = 1 + a_7(u,\xi,\eta)^2
\end{aligned}
\tag{3.14}
$$

and then $m_1(\eta) = 0$ because of the first of (3.14) and the condition $m_1(\xi) + m_4(\xi) = 0$. Then (2.29) has solution

$$
a_7(u,\xi,\eta) = tan(\mu\, u + G(\xi) - G(\eta))
$$

and hence (3.13) is true.

Second we consider the sub-case of $\alpha = 0$. Then it follows from the argument of proposition 2.2 in section two that the weight functions are

$$
\begin{aligned}
&a_1(u,\xi,\eta) = A_1(\xi,\eta) cosh\, u - A_6(\xi,\eta) sinh\, u,\\
&a_4(u,\xi,\eta) = A_4(\xi,\eta) cosh\, u + A_5(\xi,\eta) sinh\, u,\\
&a_5(u,\xi,\eta) = A_5(\xi,\eta) cosh\, u + A_4(\xi,\eta) sinh\, u,\\
&a_6(u,\xi,\eta) = A_6(\xi,\eta) cosh\, u - A_1(\xi,\eta) sinh\, u,\\
&a_7(u,\xi,\eta) = A_7(\xi,\eta),
\end{aligned}
\tag{3.15}
$$

where $A_i(\xi,\eta) = a_i(0,\xi,\eta)\,(i = 1,4,5,6,7)$ are some functions with respect to colored parameters ξ, η to be determinate. It is clear that, $A_i(\xi,\eta)\,(i = 1,4,5,6,7)$ satisfy the pure the colored Yang-Baxter (1.2).

If we substituting the initial condition (1.11) into the ones obtained by putting $v = -u$ and $\lambda = \eta$ in the system of (1.10) and combining resulting equation with the free fermion condition (2.12), one can show

$$
\begin{aligned}
&a_5(u,\xi,\eta) = -a_5(-u,\eta,\xi), \qquad a_6(u,\xi,\eta) = -a_6(-u,\eta,\xi),\\
&a_7(u,\xi,\eta) = -a_7(-u,\eta,\xi), \qquad a_4(u,\xi,\eta) = a_1(-u,\eta,\xi).
\end{aligned}
\tag{3.16}
$$

It is easy by (3.16) to show that

$$
A_4(\xi,\eta) = A_1(\xi,\eta), \qquad A_6(\xi,\eta) = -A_5(\xi,\eta). \tag{3.17}
$$

As mentioned in remark 2, all formulas obtained in this section and first two sections should hold for the pure the colored Yang-Baxter equation (1.2) except those obtained by

using symmetric operation. So we still should have

$$
\begin{aligned}
&1+A_7(\xi,\eta)^2-A_1(\xi,\eta)A_4(\xi,\eta)-A_5(\xi,\eta)A_6(\xi,\eta)=0,\\
&l_7(\eta)(A_1(\xi,\eta)A_6(\xi,\eta)+A_4(\xi,\eta)A_5(\xi,\eta))=(l_5(\eta)+l_6(\eta))A_7(\xi,\eta),\\
&l_7(\eta)(A_1(\xi,\eta)^2+A_6(\xi,\eta)^2-A_4(\xi,\eta)^2-A_5(\xi,\eta)^2=4\,l_1(\eta)A_7(\xi,\eta),
\end{aligned}
\tag{3.18}
$$

where, as mentioned in remark 2, $l_i(\eta)$, $(i=1,4,5,6,7)$ mean $\frac{\partial}{\partial\eta}A_i(\xi,\eta)|_{\xi=\eta}$. Substituting (3.17) into (3.18) we see

$$
l_5(\eta)+l_6(\eta)=0,\qquad l_1(\eta)=0.
$$

Therefore, as we did for getting (2.29) we also have

$$
(\frac{\partial}{\partial\eta}A_7(\xi,\eta))^2=l_7(\eta)(1+2\,a_7(u,\xi,\eta)^2+a_7(u,\xi,\eta)^4) \tag{3.19}
$$

and then the solution for $A_i(\xi,\eta)$ is

$$
\begin{aligned}
&A_1(\xi,\eta)=A_4(\xi,\eta)=\frac{cosh(F(\xi)-F(\eta))}{cos(G(\xi)-G(\eta))},\\
&A_5(\xi,\eta)=-A_6(\xi,\eta)=\frac{sinh(F(\xi)-F(\eta))}{cos(G(\xi)-G(\eta))},\\
&A_7(\xi,\eta)=tan(G(\xi)-G(\eta)),
\end{aligned}
\tag{3.20}
$$

where F and G are two arbitrary functions of single variable. Substituting (3.20) into (3.15) one can say (3.13) is also true for the case of $\alpha=0$, only $\mu=0$.

Description above tell us that for the case of Hamiltonian coefficients $m_5(\eta)=-m_6(\eta)$ weight functions of a gauge eight-vertex type solution must be (3.13).

Finally, straightforward calculation and computer symbolic computation can verify the following theorem.

THEOREM 3.1 : Gauge eight-vertex solutions of the colored Yang-Baxter equation (1.1) are composed of (3.1), (3.8) and (3.13) and their degenerate forms (3.2), (3.10) and (3.11).

The solutions (3.1) and (3.2) are called Baxter type solutions. They are just the solutions for "zero field" eight-vertex model by Baxter [3]. The solutions (3.8), (3.13) and their degenerate forms (3.10), (3.11) satisfy free fermion condition and are called Free-Fermion type solutions.

If we take $\lambda = 1$, $G(\xi) = cosh(2\xi)$, $H(\xi) = sinh(2\xi)$ and $F(\xi) = 0$ in solution (3.8) then (3.8) will reduce to the following solution

$$a_1(u,\xi,\eta) = cosh(\xi-\eta)cd(u) + sinh(\xi+\eta)sn(u),$$

$$a_4(u,\xi,\eta) = cosh(\xi-\eta)cd(u) - sinh(\xi+\eta)sn(u),$$

$$a_5(u,\xi,\eta) = cosh(\xi+\eta)sn(u) - sinh(\xi-\eta)cd(u),$$

$$a_6(u,\xi,\eta) = cosh(\xi+\eta)sn(u) + sinh(\xi-\eta)cd(u),$$

$$a_7(u,\xi,\eta) = k\, sn(u)\, cd(u)$$

which is given in Ref. 20.

§4. General solutions

In this paper we have shown and classified all gauge eight-vertex solutions of the colored Yang-Baxter equation (1.1). These gauge solutions and trivial solutions (1.12a), (1.12b) together with five solution transformations discussed in the first section will give all eight-vertex type solutions.

If we take in (3.8) and (3.9)

$$G(\xi) = \frac{1}{sn(\xi)}, \qquad H(\xi) = \frac{cn(\xi)}{sn(\xi)}, \qquad F(\xi) = 0, \qquad \lambda = \frac{1}{2}$$

and the solution transformation B with

$$g(u,\xi,\eta) = \sqrt{e(\xi)e(\eta)sn(\xi)sn(\eta)}\,\frac{(1-e(u))}{sn(u/2)},$$

where the elliptic exponential

$$e(\zeta) = cn(\zeta) + i\, sn(\zeta)$$

then using addition theorems for elliptic functions $sn(\zeta), cn(\zeta)$ and $dn(\zeta)$ we can obtain

the following solution given in Ref. 12

$$a_1(u,\xi,\eta) = 1 - e(u)\,e(\xi)\,e(\eta),$$

$$a_2(u,\xi,\eta) = a_3(u,\xi,\eta) = \sqrt{e(\xi)e(\eta)sn(\xi)sn(\eta)}\,\frac{(1-e(u))}{sn(u/2)},$$

$$a_4(u,\xi,\eta) = e(u) - e(\xi)\,e(\eta),$$

$$a_5(u,\xi,\eta) = e(\xi) - e(u)\,e(\eta),$$

$$a_6(u,\xi,\eta) = e(\eta) - e(u)\,e(\xi),$$

$$a_7(u,\xi,\eta) = a_8(u,\xi,\eta) = -ik\,\sqrt{e(\xi)e(\eta)sn(\xi)sn(\eta)}\,(1-e(u))\,sn(u/2),$$

the detail of calculation of which is omitted.

Similarly, all non-trivial general solutions can be also classified into two types. The first are Baxter type solutions if they can be obtained via gauge Baxter solutions and some solution transformations. The second are Free-Fermion solutions if they can be obtained via gauge Free-Fermion solutions and some solution transformations.

According to the standard method by Baxter, for a given R-matrix the spin-chain Hamiltonian is generally of the following form:

$$H = \sum_{j=1}^{N}(J_x\,\sigma_j^x\sigma_{j+1}^x + J_y\,\sigma_j^y\sigma_{j+1}^y + J_z\,\sigma_j^z\sigma_{j+1}^z + \frac{1}{2}h\,(\sigma_j^z + \sigma_{j+1}^z)),$$

where σ^x, σ^y, σ^z are Pauli matrices and the coupling constants are

$$J_x = \frac{1}{4}(m_5+m_6+m_7+m_8), \quad J_y = \frac{1}{4}(m_5+m_6-m_7-m_8),$$

$$J_z = \frac{1}{4}(m_1-m_3+m_4-m_2), \quad h = \frac{1}{4}(m_1-m_3-m_4+m_2).$$

In this paper we have proved that the hamiltonian coefficients of a gauge solution must obey

$$m_1^2 = m_4^2, \qquad m_5^2 = m_6^2.$$

It follows from the solution transformations B and D that

$$(m_1-m_3)^2 = (m_4-m_2)^2, \qquad m_5^2 = m_6^2$$

for general solutions. This clearly describes the relation between classifications of eight-vertex type solutions and spin-chain Hamiltonians. For example, if $J_x+J_y=h$, $J_z=0$, i.e. a special free-fermion model in a magnetic field, then one has $m_5 = m_1-m_3 = -m_4+m_2$.

The corresponding solution of the colored Yang-Baxter equation should be (3.11). Then the transfer matrix is of trigonometric function type.

Acknowledgement: This work was carried out while the author was visiting Department of Pure Mathematics, University of Adelaide. The author would like to thank Alan Carey for his invitation to the department and the department for kind hospitality.

This work was supported by Climbing Up Project, NSCC and Natural Scientific Foundation of Chinese Academy of Sciences.

References

[1] Yang, C.N., Phys. Rev. Lett., **19** (1967), 1312-1314.

[2] Yang, C.N., Phys. Rev. Lett., **168** (1968), 1920-1923.

[3] Baxter, R.J., Ann. Phys. **70** (1972), 193-228.

[4] Zamolodchikov, A. B. and Zamolodchikov A. B., Annals of Physics, **120** (1979), 253-291.

[5] Baxter, R.J., Exactly solved models in statistical mechanics, Academic Press, London, 1982.

[6] Jimbo, M., Yang-Baxter equation in integrable systems, World Scientific, 1989.

[7] Drinfel'd, V.G., "Quantum Groups", Proceeding of th International Congress of Mathematicians, Berkeley, (1987), 798-820.

[8] Alvarez-Gaumé, L., Cómez, C. and Sierra, G., Nucl. Phys, **319** (1989) 155; ibid. B **330** (1990) 347; Phys. Lett., **220** (1989) 142; Cómez, C. and Sierra, G., Nucl. Phys., **352** (1991) 791.

[9] Frenkel, B, and Reshetikhin, N, Yu., Commun. Math, Phys. **146** (1992) 1.

[10] Turaev, V. G., Inven. Math., 92 (1988), 527.

[11] Akutsu, Y. and Wadati, M., J. Phys. Soc. Japan. **56** (1987) 839-842.

[12] Bazhanov, V. V. and Stroganov .Y. G., Theoret. Math. Fiz., **62**(1985), 253-260.

[13] M.L. Ge and K. Xue, J. of Phys. A: Math. & Gen., **26** (1993), 281.

[14] Gustav W. Delius, Mark D. Gould, Yao-Zhong Zhang, Nucl. Phys., **B432** (1994) 377.

[15] Anthony J. Bracken, Mark D. Gould, Yao-Zhong Zhang and Gustav W. Delius, J. Phys., **A27** (1994), 6551.

[16] Sun, X. D., Wang, S. K. and Wu, K., Six-vertex type solutions of the colored Yang-Baxter equation, to appear in J. of Math. Phys..

[17] Fan, C. and Wu, F. Y., Phys. Rev. **B2**(1970), 723.

[18] Murakami, J., A state model for the multi-variable Alexander polynomial 1990, preprint, Osaka University;

[19] Cuerno,R., Gómez,C., López, E. and Sierra,G., Phys. Lett. B., **307** (1993), 56-60.

[20] Murakami, J., Int. Jour. Mod. Phys., **A7** Suppl. **1b** (1992) 765.

[21] Ruiz-Altaba, M, Phys. Lett., **277** (1992) 326.

[22] Wu Wen-tsun, Scientia Sinica, **21** (1978), 157-179.

Shi-kuh Wang, Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100080, P. R. of China.

E-mail address: xyswsk@sunrise.pku.edu.cn